Department of Mathematics
Pattamundai College, Pattamundai

# Core 13, 6th Semester

## Paper : Metric Spaces and Complex Analysis

### Section–A

### Unit – 1

1. What is Pseudometric.
2. Write the cauchy-schwarz inequality.
3. What is bounded metric space.
4. $\bar{A}$ is the smallest closed super set of A. (T/F).
5. The complex number $\varphi$ is Separable. (T/F)
6. Every real valued function defined and continuous on a closed and bounded interval is uniformly continuous. (T/F)
7. The limit of a sequence, if it exists is unique. Justified your answer.
8. Write the definition of isometry of two metric spaces.
9. Write the definition of sequential characterization of continuity.
10. Write the difference between exterior and frontier points.
11. Write the difference between limit point and adherent point.
12. Find set of all adherent point of S = [0,1].

### Unit – 2

13. Show by an example homeomorphism need not preserve completeness.
14. Which of the following statement is not correct
    a) (0.1) and [1,2] are separated
    b) The closure of a connected set is connected.
    c) The real line is connected.
15. Which of the following statements is not correct.
    a) The real line is a complete metric space.
    b) The complex plane wit the usual metric is complete.
    c) The space of continuous function on [a,b] is compact.

P.T.O.

16. The set {0,1} in R with usual metric is

a) compact (b) Open (c) closed (d) connected

17. In a discrete metric space, the only connected subsets are

a) all singleton subsets (b) all subsets

c) all nonempty subset (d) all finite subsets

18. Which of the following metric space is not complete under metric $d(x,y) = |x-y|$.

a) z (b) Q (c) R (d) $\varphi$

19. Let $f : R \to R$ be a continuous map. Then $f^{-1}\{(0,1)\}$ is

a) an open set (b) a closed set (c) both

20. Let $(M_1,d_1)$ be a discrete metric space and $(M_2,d_2)$ be any metric space. Then $f: M_1 \to M_2$ is a

a) discrete function (b) Continuous function (c) Uniformly continuous function

d) discontinuous function.

21. Let $f: R \to R$, which are not uniformly continuous.

a) $f(x) = \sin x$ (b) $f(x) = x^3$ (c) $f(n) = \frac{1}{x}$ (d) $f(x) = \cos x$.

22. In R the diameter of the set {0, 1, 2, ........100} is

a) 100 (b) 99 (c) 101 (d) 0

23. Any totally bounded metric space is

a) separable (b) not bounded (c) not complete (d) not separable

24. Every concergent sequence is ____________?

## Unit – 3

25. Write the difference between the connected and pathwise connected.

26. What is domain of definition

27. If f( z ) is a complex valued function then $f(\bar{z}) = \overline{f(z)}$. (T/F).

28. $\lim\limits_{z \to z_0} f(z) = \infty \iff \lim\limits_{z \to z_0} \frac{1}{f(z)} = 0$. (T/F)

29. If $T = \frac{az+b}{cz+d}$, $ad-bc \neq 0$. Then show that $\lim\limits_{z \to \infty} T(z) = \infty$ if c=0

30. If $f(z) = z^3+2z^2+i$, then find $f'(z)$.

31. Write the polar form of C–R equation

32. Show that $u(x,y) = x^2-y^2$ is a harmonic function.

33. Determine the set of all singular points of $f(z) = \frac{z-1}{z^2+1}$

34. If $f(z) = u+iV$ is an analytic function then u is a harmonic conjugate of _________?

35. A complex function f(z) is said to the analytic at a point $z_0$, if it is differentiable in some neighborhood of $z_0$. (T/F)

36. Write the difference between the analytic function and holomorphic function.

**Unit – 4**

37. What is transcendental function.

38. Find the period of the function $f(z) = e^z$.

39. Find the primitive period of the function $f(z) = \sin^2 z$.

40. $-i \sin h(iz) = \sin z$( T/F)

41. Show that $\cos h(iz) = \cos z$.

42. Let f(z) be a piecewise continuous function defined on a contour $\gamma = x(t)+iy(t)$, $a \leq t \leq b$. Then $\int_\gamma f(z)\,dz$ = ________?

43. Write the statement (only) ML inequality.

44. $f(z) = \sin z$ is a bounded function. (T/F).

45. What curve is represent by the following function $(1+2i)t$, $0 \leq t \leq 4$

46. Represent the following function in the form $z = z(t)$, where $|z-3+4i| = 4$

47. What is multiply connected domains.

48. Let f(z) be analytic in $\Omega|\{p\}$ and continuous on $\Omega$. Then $\int_\gamma f(z)\,dz = 0$ Where $\gamma$ is the parametrized boundedary of a plane triangle contained in $\Omega$.

**Unit – 5**

49. Let f(z) be an entire function. If $u(x,y) = Ref(z)$ is bounded for all (x,y) in complex plane, then u(x,y) and k(x,y) are ________?

50. Write the definition of convex hull.

51. If all the zeros of the polynomial p(z) lie in a half plane then the zeros of its derivative also lie in ________.

P.T.O.

52. Let $p(z) = a_0+a_1z + a_2z^2 +.....+ anz^n$, $an\neq 0$. Then there exists a point $z_0$ in $\varphi$ such that $p(z_0) =0$. (T/F)

53. A sequence {an} of complex numbers converges to P if and only if every subsequence converges to P. (T/F).

54. If $z_1$ is a point interior to the circle of convergence |z|=R of a power series $\sum_{n=0}^{a}$ an $z^n$, then the series is uniformly convergent in the closed disc $|Z| \leq |Z_1|$. (T/F)

55. If $f(z) = \sum_{n=0}^{a} an\ z^n$ converges in |Z| < R, then derivatives of all orders exist in ________.

56. Obtain Maclaurin series expansion of the function $f(z) = e^{2z}$.

57. Obtain Maclaurin series expansion of the function f(z) = sin z.

58. The zeros of an analytic function are __________.

59. Prove that if the set p of all periods of an analytic function f has a limit point then f is constant.

60. Show that Taylors series has no principal part.

**Section – B**

**Unit – 1**

1. Let x be a non-empty set and defined a mapping d: $x \times x \rightarrow R$ as follows.

$$d(x,y) = \begin{cases} 0 & \text{when } x=y \\ 1 & \text{when } x\neq y \end{cases}$$

Then show that d is a metric on x.

2. Show that every open sphere is open set but the converse is not true.

3. Show that the intersection of a finite number of open set is open.

4. Let A be a subset of a metric space (x,d), then show that A is closed if and only if $\overline{A} = A$.

5. Let (x,d) be a metric space and A and B be arbitrary subset of X. Then show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

6. Show that a discrete metric space in spareable if and only if it is countable.

7. Show that the mapping $f : R \rightarrow [-1,1]$ defined by $f(x) = \frac{x}{1+|x|}$ $\forall x \in R$ is a homeomorphism of the real line onto the open interval (–1,1) with the usual metric.

8. Every convergent sequence is a Cauchy sequence but the converse is not necessarily true.

9. Consider the space x = [0,1] of the real line with usual metric. Let $x_n = \frac{1}{n}$ $\forall$ $n \in NI$. Then show that $<x_n>$ is a cauchy sequence.

10. Every isometry is a homeomorphism.

11. A function f from a metric space x into metric space $\Upsilon$ is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

12. If A is a subset of a metric space (x,d) then $x \sim A^0 = \overline{(X \sim A)}$

## Unit – 2

13. Show that completness is preserved under isometries.

14. Show that a complete sub space of a metric space is closed.

15. Let (x,d) be a complete metric space and y be subspace of x. Then y is complete if and only if y is closed.

16. Every closed subset of a compact metric space is compact prove it.

17. A continuous image of a compact metric space is compact.

18. Show that every compact metric space is sequentially compact.

19. Show that every compact metric space is countably compact.

20. A metric (x,d) is connected if and only if the only non-empty sub set of x which is both open and closed is x itself. Prove it.

21. Show that the connectedness is preserved under homeomorphism.

22. Show that the connectedness is preserved under isometrics.

23. Show that A sub set of real line is connected if and only if it is an interval.

24. Prove that the product of two connected space is connected.

## Unit – 3

25. Show that every bounded infinite set in the complex plane has atleast one limit point.

26. In case f(z) is given in u(x,y) + i v(x,y) and $\lim_{z \to 2+3i} \{x + i(x+y)^2\} = 2+11i$. Prove it by the method $(\in - \delta)$.

27. Show that $f(z)=z^2$ is uniformly continuous in the region $|z|<1$, but the function $g(z) = \frac{1}{z}$ is not uniformly continuous in this region.

28. Show that the function $f(z) = |z|^2$ is continuous in the whole complex plane but not differentiable in $\varphi$ except origin.

29. Show that the function $f(z) = \frac{1}{z-1}$ is analytic at $z=1+i$.

30. Prove that f(z) = sinx coshy – i cos x sinhy is no where analytic.

31. Prove that the linear combination $c_1 f_1(z) + c_2 f_2(z)$ of two entire functions $f_1(z)$ and $f_2(z)$ is again an entire function, where $c_1$ and $c_2$ are complex constants.

P.T.O.

32. State and prove the complex form of C–R equations.

33. Let f(z) = u (x,y) + i v(x,y) be analytic. Then show that f(z) = 2u $\left(\frac{z}{2}, -\frac{iz}{2}\right)$ + constant.

34. Show that the function $u(x,y) = -x^3 + 3xy^2 + 2y + 1$ is harmonic in $\varphi$. Find the harmonic conjugate.

35. For what values of 'a' the function $e^{ax}$ cosy is harmonic.

36. If v is harmonic conjugate of u, then u may not be harmonic conjugate of v. Support your argument by giving some example.

## Unit – 4

37. Show that the equation $e^z = w$ has infinitely many solutions.

38. Solve the equation $e^{2z-1} = 1 + i$.

39. Show that $\exp(i\bar{z}) = \overline{\exp(iz)}$ if and only if $z = n\pi, n \in z$.

40. Find all zeros and period of the function cos (2iz +13)

41. Prove that $\int_{\Upsilon} f(z)dz = \int_{\Upsilon_1} f(z)dx + \int_{\Upsilon_2} f(z)\, dz$, where $\Upsilon = \Upsilon_1 + \Upsilon_2$.

42. Find the upper bound of the integral (without evaluating the integral) $\int_c \left|\frac{(z^2+3)\, e^{iz}\, \mathrm{Log}\, z}{z^2-2}\, dz\right|$,

where $C = \{ z : z = 2e^{i\theta}, 0 < \theta < \frac{\pi}{3}\}$.

43. Show that if a function f(z) is analytic in a simply connected domain D, then $\int_{\Upsilon} f(z)dx = 0$ for every closed contour $\Upsilon$ lying in D.

44. Use the principle of deformation to show that for $n \in I$, $\int_{\Upsilon} \frac{1}{(z-z_0)^n}\, dz = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases}$

45. Evaluate the integral $\int_{-1-i}^{1-i} \frac{1}{z} dz$, along the curve $|z| = \sqrt{2}$ having negative orientation.

46. With the help of the Cauchy integral formula, evaluate the integral $\int_{|z|=1} \frac{\cos(2\pi z)}{(2z-1)(z-3)}\, dz$.

47. Compute $\int_{\Upsilon} \frac{e^{i\pi z}(z+i)^2 \cos nz}{z^2-1}\, dz$, where $\Upsilon = \{ z : |z| = 2\cos\theta : -\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2} \}$

48. If a function f(z) is continuous throughout a domain D and $\int_{\Upsilon} f(z)dz = 0$, for every closed contour $\Upsilon$ in D, then f(z) is analytic throughout $\theta$.

## Unit – 5

49. A non constant entire function in unbounded.

50. Suppose f(z) and g(z) are entire functions, g(z) is never zero and $|f(z)| \leq |g(z)|$ for all z. Show that there is a constant c such that f(z) = cg(z).

51. A Polynomial of degree n has exactly n zezos counting multiplicity.

52. Let P(z) be a polynomial of degree $n \geq 1$. Then every zero of $P^{|}(z)$ also lies in the convex hull of the set of zeros of p(z).

53. Let f(z) be an entire function which satisfies the condition for all $z \in \varphi$, Ref(z) or Imf(z) as no zeros. Then prove that f(z) is constant.

54. Let $\sum_{n=1}^{\infty}$ (an) be an absolutely conversent series having sum S. Then prove that , every rearrangement of $\sum_{n=1}^{\infty}$ (an) also converses absolutely and has the sum S.

55. Discuss the convergence of the series $\sum_{n=1}^{\infty}$ an, where $a_{2n-1} = \left(\frac{1+i}{3\sqrt{2}}\right)^{2n-1}$ and $a_{2n} = \left(\frac{1-i}{2\sqrt{2}}\right)^{2n}$

56. Find the radius of convergence of the series

i) $\sum_{n=1}^{\infty} n^n z^n$ (ii) $\sum_{n=1}^{\infty} \frac{z^{2^n}}{2^n}$

57. Write series expansions for sin (hz).

58. Derive the Laurent series of the function $f(z) = z^3 \cos(h\frac{1}{z})$.

59. Expand $f(z) = \frac{1}{z^2+4}$ in powers of (z+2i)

60. Find the principal part of the function $f(z) = \frac{1}{z^2(1-z)}$

## Section – C

## Unit – 1

1. Let (x,d) be a metric space, and let $d^*(x,y) = \frac{d(x,y)}{1+d(x,y)}$, for all x,y in x. Then prove that d* is a bounded metric on x which is equivalent to d.

2. Let $(x,d_1)$ and $(y,d_2)$ be two metric spaces. A functionh $f: x \to y$ is continuous on X iff for each open set $G \subset y$, $f^{-1}(G)$ is an open subset of x.

3. Show that the real line is a complete metric space.

P.T.O.

4. Let $(x,d_1)$ and $(y, d_2)$ be two metric spaces. A sequence $<x_n, y_n>$ in the product space converges to (x,y) if and only if $x_n \to x$ and $y_n \to y$

5. Prove that the space of C [0,1] of all continuous real valued function on [0,1] with metric d defined by d (f,g) = sup {|f(x)–g(x)|: $x \in [0,1]$} is a complete metric space.

## Unit – 2

6. If <Fn> is a nested sequence of non-empty closed sub sets of metric space (x,d) such that $\delta(Fn) \to 0$ as $n \to \infty$ then x is complete if and only if $\bigcap_{n=1}^{\infty} F_n$ consists of exactly only one point, where $\delta(Fn)$ denoted the diameter of Fn.

7. Every contraction mapping T on a complete metric space (x,d) has a unique fixed point.

8. Show that every complete metric space is of second category.

9. A metric space is sequentially compact if and only if it has the Bolzano- weierstrass property.

10. Let (x,d) we a metric space and let E be a connected subset of X. If F is a subset of x such that $E \subset F \subset \overline{E}$, then F is connected In particular, $\overline{E}$ is connected.

## Unit – 3

11. If a function f is continuous in a domain D, Prove that the functions |f(z)|, $f(\overline{z})$ and $\overline{f(z)}$ are also continuous in the domain

12. State and prove Cauchy-Riemann equations.

13. For the function f(z) defined by $f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z} & , z \neq 0 \\ 0 & z = 0 \end{cases}$ show that the C–R equations are satisfied at (0,0) but the function is not differentiable at 0 + i0

14. Find an analytic function whose real part is u(x,y) = 2xy+2x.

15. Let f(z) be an analytic function defined in a domain D, and Re(f(z)) =u is a constant. Then prove that f(z) must be constant.

## Unit – 4

16. If Log sin $(\theta+i\phi) = \alpha+i\beta$, show that $2 \cos 2\theta = e^{2\phi} + e^{-2\phi} - 4e^{2\alpha}$.

17. Integrate the functions $f(z) = x+y-3ix^2$ and $g(z) = x+(2+y)i$ between the two points z =0 and z = 1+i along differed paths.

18. Suppose that a function f is continuous in a domain D. Then the following conditions are equivalent.

a) f has antiderivative F in D.

b) The integrals of f(z) along any path between any two fixed points in D is independent of path.

c) The integral of f(z) along every closed contour in D is zero.

19. Let f(z) be analytic in a domain $\Omega$ and $\Upsilon$ be a simple closed contour in $\Omega$, taken in a positive sense. Then for all points z interior to $\Upsilon$, then show that $f'(z) = \frac{1}{2\pi i} \int_{\Upsilon} \frac{f(s)}{(S-z)^2} ds$.

20. Let f be analytic in a domain D and $f'(z_0) \neq 0$ for some $z_0 \in D$. Then there exists a neighbourhood of $z_0$ on which f is one–one.

## Unit – 5

21. State and prove Liouvilles theorem.

22. Suppose that f(z) is analytic throughout a domain $\Omega$ and let a be any point in $\Omega$. Then, for all z:$|z-a| < R \leq \Omega$, f(z) has the series representation $f(z) = \sum_{n=0}^{\infty} an(z-a)^n$, $|z-a| < R$, where $an = \frac{1}{2\pi i} \int_{\Upsilon} \frac{f(s)}{(S-a)^{n+1}} ds = \frac{f^n(a)}{n!}$, n =0,1, 2... and $\Upsilon$ is any positively oriented circular path with centre of a and radius r<R.

23.a) Obtain the Laurent series expansion of $f(z) = \frac{z}{z^2 - 4z + 3}$

b) Expand the function of f(z) in the power of (z–1) and mention the regions of validity.

24. If the series $\sum_{n=-\infty}^{\infty} a_n(Z-Z_0)^n$ converges to f(z) in an annular region $R_1<|z-z_0|< R_2$, then it is the Laurent expansion of f(z) about $z = z_0$.

25. Prove that the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly in any closed disc $|z| \leq r<1$. However, it is not uniformly convergent in its whole disc of convergence $|z| <1$.

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